

NONUNIFORMLY EXPANDING 1D MAPS WITH LOGARITHMIC SINGULARITIES

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ABSTRACT. For a certain parametrized family of maps on the circle with critical points and logarithmic singularities where derivatives blow up to infinity, we construct a positive measure set of parameters corresponding to maps which exhibit nonuniformly expanding behavior. This implies the existence of “chaotic” dynamics in dissipative homoclinic tangles in periodically perturbed differential equations.

1. INTRODUCTION

Let $f_a : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$(1) \quad f_a : x \mapsto x + a + L \cdot \ln |\Phi(x)|, \quad L > 0,$$

where $a \in [0, 1]$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 satisfying: (i) $\Phi(x+1) = \Phi(x)$; (ii) $\Phi'(x) \neq 0$ if $\Phi(x) = 0$, (iii) $\Phi''(x) \neq 0$ if $\Phi'(x) = 0$. The family (f_a) induces a parametrized family of maps from $S^1 = \mathbb{R}/\mathbb{Z}$ to itself. In this paper we study the abundance of nonuniform hyperbolicity in this family of circle maps.

Our study of (f_a) is motivated by the recent studies of [15, 16, 17, 18] on homoclinic tangles and strange attractors in periodically perturbed differential equations. When a homoclinic solution of a dissipative saddle is periodically perturbed, the perturbation either pulls the stable and the unstable manifold of the saddle fix point completely apart, or it creates chaos through homoclinic intersections. In both cases, the separatrix map induced by the solutions of the perturbed equation in the extended phase space is a family of two-dimensional maps. Taking a singular limit, one obtains a family of one-dimensional in the form of (1) (with the absolute value sign around $\Phi(x)$ removed). Let μ be a small parameter representing the magnitude of the perturbation and ω be the forcing frequency. We have $a \sim \omega \ln \mu^{-1} \pmod{1}$, $L \sim \omega$; and Φ is the classical Melnikov function (See [16, 17, 18]).

When we start with *two* unperturbed homoclinic loops and assume symmetry, then the separatrix maps are a family of annulus maps, the singular limit of which is precisely f_a in (1) (See [15]). If the stable and unstable manifolds of the perturbed saddle are pulled completely apart by the forcing function, then $\Phi(x) \neq 0$ for all x . In this case we obtain strange attractors, to which the theory of rank one maps developed in [21] apply. If the stable and unstable manifold intersect, then $\Phi(x) = 0$ is allowed and the strange attractors are associated to homoclinic intersections. For the modern theory of chaos and dynamical systems, this is a case of historical and practical importance; see [3, 9, 10]. To this case, unfortunately, the theory of rank one maps in [21] does not apply because of the existence of

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the singularities of f_a . Our ultimate goal is to develop a theory that can be applied to the separatrix maps allowing $\Phi(x) = 0$. This paper is the first step, in which we develop a 1D theory.

For $f = f_a$, let $C(f) = \{f'(x) = 0\}$ be the set of critical points and $S(f) = \{\Phi(x) = 0\}$ be the set of singular points. In this paper we are interested in the case $L \gg 1$. As L gets larger, the contracting region gets smaller and the dynamics is more and more expanding in most of the phase space. Nevertheless, the recurrence of the critical points is inevitable, and thus infinitesimal changes of dynamics occur when a is varied. In addition, the logarithmic nature of the singular set S turns out to present a new phenomenon [11] which is unknown to occur for smooth one-dimensional maps with critical points.

Our main result states that nonuniform expansion prevails for “most” parameters, provided $L \gg 1$. Let $\lambda = 10^{-3}$ and let $|\cdot|$ denote the one-dimensional Lebesgue measure.

Theorem. *For all large L there exists a set $\Delta = \Delta(L) \subset [0, 1)$ of a -values with $|\Delta| > 0$ such that if $a \in \Delta$ then for $f = f_a$ and each $c \in C$, $|(f^n)'(fc)| \geq L^{\lambda n}$ holds for every $n \geq 0$. In addition, $|\Delta| \rightarrow 1$ holds as $L \rightarrow \infty$.*

For the maps corresponding to the parameters in Δ , our argument shows a nonuniform expansion, i.e, for Lebesgue a.e. $x \in S^1$, $\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln |(f^n)'x| \geq \frac{1}{3} \ln L$. In addition, combining our argument with an argument in [[19] Sect.3] one can construct invariant probability measures absolutely continuous with respect to Lebesgue measure (acips). The main difference from the smooth case is to bound distortions, which can be handled with Lemma 2.2 in this paper. A careful construction exploiting the largeness of L shows the uniqueness of acips and some advanced statistical limit theorems (See [11]).

Since the pioneering work of Jakobson [4], there have been quite a few number of papers over the last thirty years dedicated to proving the abundance of chaotic dynamics in increasingly general families of smooth one-dimensional maps [1, 2, 8, 12, 13, 14, 19]. Families of maps with critical and singular sets were studied in [5, 6, 7]. One key aspect of the singularities of our maps that has no analogue in [5, 6, 7] is that, returns to a neighborhood of the singular set can happen very frequently. The previous arguments seem not sufficient to deal with points like this. To avoid problems arising from the logarithmic singularities, and to get the asymptotic estimate on the measure of Δ , we introduce new arguments:

- our definition of bound periods (see Sect.2.4) incorporates the recurrence pattern of the critical orbits to both C and S . Thus, the resultant bound period partition depends on the parameter a , and is not a fixed partition, as is the case in [1, 2];
- to get the asymptotic estimate $|\Delta| \rightarrow 1$ as $L \rightarrow 1$, we need to abandon starting an inductive construction with small intervals around Misiurewicz parameters. Instead we start with a large parameter set (denoted by Δ_N), which is a union of a finite number of intervals. This necessitates additional works on establishing uniform hyperbolicity outside of a neighborhood of C , which is rather straightforward around Misiurewicz parameters.

The rest of this paper consists of two sections. In Sect.2 we perform phase space analyses. Building on them, in Sect.3 we construct the parameter set Δ by induction. To estimate the measure of the set of parameters excluded at each step, instead of Benedicks & Carleson [1, 2] we elect to follow the approach of Tsujii [13, 14], primarily because partitions depend on a , and the extension of this approach is more transparent in our dealing with the issues

related to the singularities. Unlike [1, 2], the current strategy relies on a geometric structure of the set of parameters excluded at each step. In addition, there is no longer the need for a large deviation argument, introduced originally in [2] as an independent step of parameter exclusions.

2. PHASE SPACE ANALYSIS

In this section we carry out a phase space analysis. Elementary facts on f_a are introduced in Sect. 2.1. In Sect.2.2 we prove a statement on distortion. In Sect.2.3 we discuss an initial set-up. In Sect.2.4 we introduce three conditions, which will be taken as assumptions of induction for the construction of the parameter set Δ , and develop a binding argument. In Sect.2.5 we study global dynamical properties of maps satisfying these conditions.

2.1. Elementary facts. For $\varepsilon > 0$, we use C_ε and S_ε to denote the ε -neighborhoods of C and S respectively. The distances from $x \in S^1$ to C and S are denoted as $d_C(x)$ and $d_S(x)$ respectively. We take L as a base of $\log(\cdot)$.

Lemma 2.1. *There exist $K_0 > 1$ and $\varepsilon_0 > 0$ such that the following holds for all sufficiently large L and $f = f_a$:*

(a) *for all $x \in S^1$,*

$$K_0^{-1}L \frac{d_C(x)}{d_S(x)} \leq |f'x| \leq K_0L \frac{d_C(x)}{d_S(x)}, \quad |f''x| \leq \frac{K_0L}{d_S^2(x)};$$

(b) *for all $\varepsilon > 0$ and $x \notin C_\varepsilon$, $|f'x| \geq K_0^{-1}L\varepsilon$;*

(c) *for all $x \in C_{\varepsilon_0}$, $K_0^{-1}L < |f''x| < K_0L$.*

Proof. This lemma follows immediately from

$$f' = 1 + L \cdot \frac{\Phi'}{\Phi}; \quad f'' = L \cdot \frac{\Phi''\Phi - (\Phi')^2}{\Phi^2}$$

and our assumptions on Φ in the beginning of the introduction. \square

2.2. Bounded distortion. Let $c \in C$, $c_0 = fc$, and $n \geq 1$. Let

$$(2) \quad D_n(c_0) = \frac{1}{\sqrt{L}} \cdot \left[\sum_{i=0}^{n-1} d_i^{-1}(c_0) \right]^{-1} \quad \text{where} \quad d_i(c_0) = \frac{d_C(c_i) \cdot d_S(c_i)}{J^i(c_0)}.$$

Lemma 2.2. *For all $x, y \in [c_0 - D_n(c_0), c_0 + D_n(c_0)]$ we have $J^n(x) \leq 2J^n(y)$, provided that $c_i \notin C \cup S$ for every $0 \leq i < n$.*

Proof. Write D_n for $D_n(c_0)$, and let $I = [c_0 - D_n, c_0 + D_n]$. Then

$$\log \frac{J^n(x)}{J^n(y)} = \sum_{j=0}^{n-1} \log \frac{J(f^j x)}{J(f^j y)} \leq \sum_{j=0}^{n-1} |f^j I| \sup_{\phi \in f^j I} \frac{|f''\phi|}{|f'\phi|}.$$

Lemma 2.2 would hold if for all $j \leq n-1$ we have $f^j I \cap (S \cup C) = \emptyset$ and

$$(3) \quad |f^j I| \sup_{\phi \in f^j I} \frac{|f''\phi|}{|f'\phi|} \leq \log 2 \cdot d_j^{-1}(c_0) \left[\sum_{i=0}^{n-1} d_i^{-1}(c_0) \right]^{-1}.$$

We prove (3) by induction on j . Assume (3) holds for all $j < k$. Summing (3) over $j = 0, 1, \dots, k-1$ implies $\frac{1}{2} \leq \frac{J^k(\eta)}{J^k(c_0)} \leq 2$ for all $\eta \in I$. We have

$$(4) \quad |f^k I| \leq 2J^k(c_0)D_n = 2d_k^{-1} \cdot d_C(c_k)d_S(c_k)D_n \leq \frac{2}{\sqrt{L}}d_C(c_k)d_S(c_k).$$

We have $f^k I \cap (C \cup S) = \emptyset$ from (4), and for $\phi \in f^k I$,

$$\begin{aligned} |f^k I| \frac{|f''\phi|}{|f'\phi|} &\leq 2d_k^{-1}d_C(c_k)d_S(c_k)D_n \cdot \frac{K_0^2}{d_C(\phi)d_S(\phi)} \\ &= 2K_0^2d_k^{-1}D_n \cdot \frac{d_C(c_k)d_S(c_k)}{d_C(\phi)d_S(\phi)} \leq \frac{2K_0^2}{\sqrt{L}} \cdot d_k^{-1} \left[\sum_{i=0}^{n-1} d_i^{-1} \right]^{-1}, \end{aligned}$$

where we used Lemma 2.1(a) for $\frac{|f''\phi|}{|f'\phi|}$ for the first inequality. For the last inequality we observe that the second factor of the left-hand-side is < 2 by (4). \square

2.3. Initial setup. In one-dimensional dynamics, a general strategy for constructing positive measure sets of “good” parameters is to start an inductive construction in small parameter intervals, in which orbits of critical points are kept out of bad sets for certain number of iterates. One way to find these intervals is to first look for Misiuriewicz parameters, for which all critical orbits stay out of the bad sets under any positive iterate. We would then confine ourselves in small parameter intervals containing the Misiuriewicz parameters, and would eventually prove that the Misiuriewicz parameters are Lebesgue density points of the good parameter sets. This approach for initial set-ups, however, is with some drawbacks. First, for a one-parameter family of maps with multiple critical points, the Misiuriewicz parameters are relatively hard to find because of the need of controlling multiple critical orbits with one parameter. Though the argument in [20] is readily extended to cover our family, we are nevertheless up to a hard start. Second, with the rest of the study confined in a small parameter interval containing a Misiuriewicz parameter, it is not clear how we could prove the global asymptotic measure estimate ($|\Delta| \rightarrow 1$ as $L \rightarrow \infty$) of the theorem.

An alternative route that is made possible by the approach of this paper is to start with a rather straight forward and relatively weak assumption. Let $\sigma = L^{-\frac{1}{6}}$. Let N be a large integer independent of L . For $0 \leq n \leq N$, let

$$\Delta_n = \{a \in [0, 1) : f_a^{i+1}(C) \cap (C_\sigma \cup S_\sigma) = \emptyset \text{ for every } 0 \leq i \leq n\}.$$

Observe that Δ_n is a union of intervals. We start with the following statement, the proof of which is given in Appendix.

Lemma 2.3. *For any large integer N there exists $L_0 = L_0(N) \gg 1$ such that if $L \geq L_0$, then $|\Delta_N| \geq 1 - L^{-\frac{1}{9}}$.*

Lemma 2.3 is sufficient for us to move forward. This approach for initial setups is easier, and leads to the desired asymptotic measure estimate on Δ as $L \rightarrow \infty$.

We move to the expanding property of the maps corresponding to parameters in Δ_N . We frequently use the following notation: for $c \in C$ and $n \geq 1$, $c_0 = fc$ and $c_n = f^n c_0$; for $x \in S^1$ and $n \geq 1$, $J(x) = |f'x|$ and $J^n(x) = J(x)J(fx) \cdots J(f^{n-1}x)$.

Let $\alpha = 10^{-6}$ and $\delta = L^{-\alpha N}$. In what follows, we suppose N to be a large integer for which $\delta \ll \sigma$, and the conclusion of Lemma 2.3 holds. The value of N will be replaced if necessary,

but only a finite number of times. The letter K will be used to denote generic constants which are independent of N and L .

The next lemma, the proof of which is given in Appendix, ensures an exponential growth of derivatives for orbit segments lying outside of C_δ .

Lemma 2.4. *There exists $L_1 \geq L_0$ such that if $L \geq L_1$ and $f = f_a$ is such that $a \in \Delta_N(L)$, then the following holds:*

- (a) *if $n \geq 1$ and $x, fx, \dots, f^{n-1}x \notin C_\delta$, then $J^n(x) \geq \delta L^{2\lambda n}$;*
- (b) *if moreover $f^n x \in C_\delta$, then $J^n(x) \geq L^{2\lambda n}$.*

Standing assumption for the rest of this section: $L \geq L_1$ and $a \in \Delta_N$.

2.4. Recovering expansion. For $f = f_a$, $c \in C$ and $n > N$ we introduce three conditions:

- (G1) $_{n,c}$ $J^{j-i}(c_i) \geq L \min\{\sigma, L^{-\alpha i}\} \quad 0 \leq \forall i < \forall j \leq n+1$;
- (G2) $_{n,c}$ $J^i(c_0) \geq L^{\lambda i} \quad 0 < \forall i \leq n+1$;
- (G3) $_{n,c}$ $d_S(c_i) \geq L^{-4\alpha i} \quad N \leq \forall i \leq n$.

We say f satisfies (G1) $_n$ if (G1) $_{n,c}$ holds for every $c \in C$. The definitions of (G2) $_n$, (G3) $_n$ are analogous. These conditions are taken as inductive assumptions in the construction of the parameter set Δ .

We establish a recovery estimate of expansion. Let $c \in C$, $c_0 = fc$ and assume that (G1) $_{n,c}$ -(G3) $_{n,c}$. For $p \in [2, n]$, let

$$I_p(c) = \begin{cases} f^{-1}[c_0 + D_{p-1}(c_0), c_0 + D_p(c_0)] & \text{if } c \text{ is a local minima of } x \rightarrow x + a + L \cdot \ln |\Phi(x)|; \\ f^{-1}(c_0 - D_p(c_0), c_0 - D_{p-1}(c_0)) & \text{if } c \text{ is a local maxima of } x \rightarrow x + a + L \cdot \ln |\Phi(x)|. \end{cases}$$

By the non-degeneracy of c , $I_p(c)$ is the union of two intervals, one at the right of c and the other at the left. According to Lemma 2.2, if $x \in I_p(c)$ then the derivatives along the orbit of fx shadow that of the orbit of c_0 for $p-1$ iterates. We regard the orbit of x as been bound to the critical orbit of c up to time p ; and we call p the *bound period* of x to c .

Lemma 2.5. *If (G1) $_{n,c}$ -(G3) $_{n,c}$ holds, then for $p \in [2, n]$ and $x \in I_p(c)$ we have:*

- (a) $p \leq \log |c - x|^{-\frac{2}{\lambda}}$;
- (b) *if $x \in C_\delta$, then $|J^p(x)| \geq |c - x|^{-1 + \frac{16\alpha}{\lambda}} \geq L^{\frac{\lambda}{3}p}$.*

Proof. By definition we have

$$|c - x|^2 \leq D_{p-1}(c_0) \leq L^{-\frac{1}{2}} d_{p-2}(c_0) < L^{-\frac{1}{2}} J^{p-2}(c_0)^{-1}.$$

Then by (G2),

$$(5) \quad |c - x|^2 \leq L^{-\frac{1}{2} - \lambda(p-2)} \leq L^{-\lambda p},$$

from which (a) follows. The second inequality of (b) follows from (5).

Sublemma 2.6. *For $0 \leq i \leq n$ we have:*

- (a) $d_C(c_i) \geq K^{-1} \sigma L^{-\alpha i}$;
- (b) $J^i(c_0) D_{i+1}(c_0) \geq L^{-1-7\alpha i}$.

We finish the proof of Lemma 2.5 assuming the conclusion of this sublemma. We have

$$J^p(x) = J^{p-1}(fx)J(x) \geq K^{-1}J^{p-1}(c_0) \cdot L|c-x| \geq K^{-1}J^{p-1}(c_0) \cdot |c-x|^{-1}D_p(c_0),$$

where for the first inequality we use Lemma 2.2 and Lemma 2.1(c), and for the last inequality we use $x \in I_{-p}(c) \cup I_p(c)$. Using Sublemma 2.6(b) for $i = p-1$ we obtain

$$J^p(x) \geq K^{-1}L^{-1-7\alpha p}|c-x|^{-1}.$$

Substituting into this the upper estimate of p in Lemma 2.5(a) we obtain

$$J^p(x) \geq K^{-1}L^{-1}|c-x|^{-1+\frac{15\alpha}{\lambda}} \geq |c-x|^{-1+\frac{16\alpha}{\lambda}}.$$

We have used $|c-x| \leq \delta = L^{-\alpha N_0}$ for the last inequality.

It is left to prove the sublemma. (G1) implies $|f'c_i| \geq L \min\{\sigma, L^{-\alpha i}\}$. Then (a) follows from Lemma 2.1(a). As for (b), let $j \in [0, i]$. By definition,

$$J^i(c_0)d_j(c_0) = \frac{J^i(c_0)}{J^j(c_0)}d_C(c_j)d_S(c_j).$$

We have: $\frac{J^i(c_0)}{J^j(c_0)} \geq L^{-\alpha j}\sigma$ from (G1); $d_C(v_j) \geq K^{-1}\sigma L^{-\alpha j}$ from (a); $d_S(v_j) \geq \sigma L^{-4\alpha j}$ from (G3). Hence, $J^i(c_0)d_j \geq K^{-1}\sigma^3 L^{-6\alpha j}$, and thus

$$\sum_{j=0}^i J^i(c_0)^{-1}d_j^{-1}(c_0) \leq \sigma^{-3}L^{7\alpha i}.$$

Taking reciprocals implies (b). □

2.5. Global dynamical properties. At step n of induction, we wish to exclude all parameters for which one of $(G1-3)_n$ is violated for some $c \in C$, and to estimate the measure of the parameters deleted. Conditions (G1) (G2), however, can not be used directly as rules for exclusion, since they do not care about cumulative effects of “shallow returns”. Hence we introduce a stronger condition, based on the notion of *deep returns*, and will use it as a rule for deletion in Section 3.

Hypothesis in Sect.2.5: $f = f_a$ is such that $a \in \Delta_{N_0}$. $n \geq N_0$ and $(G1)_{n-1}$ -(G3) $_{n-1}$ hold for all $c \in C$.

For all $c \in C$ we have:

- (i) $f^{i+1}c \notin C \cup S$ for all $0 \leq i \leq n$;
- (ii) for the orbit of $c_0 = fc$, the bound period initiated at all returns to C_δ before n is $\ll n$.

(Bound/free structure) We divide the orbit of c_0 into alternative bound/free segments as follows. Let n_1 be the smallest $j \geq 0$ such that $c_j \in C_\delta$. For $k > 1$, we define free return times n_k inductively as follows. Let p_{k-1} be the bound period of $c_{n_{k-1}}$, and let n_k be the smallest $j \geq n_{k-1} + p_{k-1}$ such that $c_j \in C_\delta$. We decompose the orbit of c_0 into bound segments corresponding to time intervals $(n_k, n_k + p_k)$ and free segments corresponding to time intervals $[n_k + p_k, n_{k+1}]$. The times n_k are the *free return* times. We have

$$n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \dots$$

Definition 2.7. Let $c \in C$ and assume that $c_0 = fc$ makes a free return to C_δ at time $\nu \leq n$. We say ν is a *deep return* of c_0 if for every free return $i \in [0, \nu - 1]$;

$$(6) \quad \sum_{\substack{j \in [i+1, \nu] \\ \text{free return}}} 2 \log d_C(c_j) \leq \log d_C(c_i).$$

We say $(R)_{n,c}$ holds if

$$\prod_{i \in [0, j]: \text{ deep return}} d_C(c_i) \geq L^{-\frac{1}{20}\lambda\alpha j} \quad \text{for every } N_0 \leq j \leq n.$$

We say $(R)_n$ holds if $(R)_{n,c}$ holds for every $c \in C$.

Lemma 2.8. *If $(R)_{n,c}$ holds, then $(G1)_{n,c}$, $(G2)_{n,c}$ hold.*

Proof. The first step is to show

$$(7) \quad \prod_{\substack{N_0 < i \leq n \\ \text{free return}}} d_C(c_i) \geq L^{-\alpha n}.$$

We call a free return *shallow* if it is not a deep free return. Let $\mu \in (0, n)$ be a shallow free return time, and $i(\mu)$ be the largest deep free return time $< \mu$. We claim that

$$(8) \quad \sum_{\substack{i(\mu)+1 \leq j \leq \mu \\ \text{free return}}} \log d_C(c_j) \geq \log d_C(c_{i(\mu)}).$$

We finish the proof of (7) assuming (8). Let μ_1 be the largest free shallow return time in $(0, n]$, and i_1 be the largest deep free return time $< \mu_1$. We then let μ_2 be the largest shallow free return time $< i_1$, and i_2 be the largest deep free return time $< \mu_2$, and so on. We obtain a sequence of deep free return times $i_1 > i_2 > \dots > i_q$, and we have

$$(9) \quad \sum_{\substack{0 \leq j \leq n \\ \text{shallow return}}} \log d_C(c_j) \geq \sum_{j=1}^q \log d_C(c_{i_j}) \geq \sum_{\substack{0 \leq i \leq k \\ \text{deep return}}} \log d_C(c_i)$$

where the first inequality is from (8). We then have

$$\sum_{0 \leq j \leq n: \text{ free return}} \log d_C(c_j) \geq 2 \sum_{\substack{0 \leq j \leq n \\ \text{free return}}} \log d_C(c_j) \geq \alpha n,$$

where the last inequality is from $(R)_{n,c}$.

To prove (8), we let β_1 be the smallest free return time $\leq \mu - 1$ such that

$$(10) \quad \sum_{\substack{\beta_1+1 \leq j \leq \mu \\ \text{free return}}} 2 \log d_C(c_j) > \log d_C(c_{\beta_1}).$$

We claim that no deep free return occurs during the period $[\beta_1 + 1, \mu]$. This is because if $i' \in [\beta_1 + 1, \mu]$ is a deep return, then we must have

$$\sum_{\substack{\beta_1+1 \leq j \leq \mu \\ \text{free return}}} 2 \log d_C(c_j) \leq \sum_{\substack{\beta_1+1 \leq j \leq i' \\ \text{free return}}} 2 \log d_C(c_j) \leq \log d_C(c_{\beta_1}),$$

contradicting (10). If β_1 is a deep return, we are done. Otherwise we find a $\beta_2 < \beta_1$ so that

$$(11) \quad \sum_{\substack{\beta_2+1 \leq j \leq \beta_1 \\ \text{free return}}} 2 \log d_C(c_j) > \log d_C(c_{\beta_2}),$$

and so on. This process will end at a deep free return time, which we denote as $\beta_q := i(\mu)$. (8) follows from adding (10), (11) and so on up to the time for $\beta_q = i(\mu)$.

We first prove $(G1)_{n,c}$. Let $0 \leq i < j \leq n+1$. Observe that the bound periods for all returns to C_δ for the orbit of c_0 up to time n is $\leq \frac{2\alpha}{\lambda}n \ll n$. This follows from (7) and Lemma 2.5(a). Hence, it is possible to introduce the bound-free structure starting from c_i to c_j . We consider the following two cases separately.

Case I: j is free. For free segments we use Lemma 2.4, and for bound segments we use Lemma 2.5(c). We obtain exponential growth of derivatives from time i to j , which is much better than what is asserted by $(G1)_{n,c}$.

Case II: j is bound. Let \hat{j} denote the free return with a bound period p such that $j \in [\hat{j}+1, \hat{j}+p]$. We have $\hat{j} \leq n$, for otherwise $j > n+1$. Consequently,

$$J^{j-i}(c_i) = \frac{J^{\hat{j}}(c_0)}{J^i(c_0)} \cdot \frac{J^{\hat{j}+1}(c_0)}{J^{\hat{j}}(c_0)} \cdot \frac{J^j(c_0)}{J^{\hat{j}+1}(c_0)} > L^{\frac{1}{3}\lambda(\hat{j}-i)} \cdot K^{-1} L d_C(v_{\hat{j}}) \cdot K^{-1} L^{\lambda(j-\hat{j}-1)}$$

where for the last inequality, we use Lemma 2.5(c) combined with Lemma 2.4 for the first factor. For the third factor we use bound distortion and $(G2)_{n-1}$ for the binding critical orbit. It then follows that $J^{j-i}(c_0) \geq L^{\alpha(j-i)-\alpha\hat{j}} \geq L^{-\alpha i}$. Hence $(G1)_{n,c}$ holds.

As for $(G2)_{n,c}$, we introduce the bound-free structure starting from c_0 to c_{n+1} . Observe that the sum of the lengths of all bound periods for the orbit of c_0 up to time n is $\leq \frac{2\alpha}{\lambda}n \ll n$. This follows from (7) and Lemma 2.5(a). Using Lemma 2.5(b) for each bound segment and Lemma 2.4 for each free segment in between two consecutive free returns, we have

$$J^{n+1}(c_0) \geq \delta L^{2\lambda n(1-\frac{\alpha}{\lambda})} \geq L^{\lambda n}.$$

This completes the proof of Lemma 2.8. \square

The next expansion estimate at deep return times will be used in a crucial way in the construction of the parameter set Δ .

Lemma 2.9. *If $c \in C$ and $\nu \leq n+1$ is a deep return time of $c_0 = fc$, then*

$$J^\nu(c_0) \cdot D_\nu(c_0) \geq \sqrt{d_C(c_\nu)}.$$

Proof. Let $0 < n_1 < \dots < n_t < \nu$ denote all free returns in the first ν iterates of c_0 , with p_1, \dots, p_t the corresponding bound periods. Let

$$\Theta_{n_k} = \sum_{i=n_k}^{n_k+p_k-1} d_i^{-1}(c_0) \quad \text{and} \quad \Theta_0 = \sum_{i=0}^{\nu-1} d_i^{-1}(c_0) - \sum_{k=1}^t \Theta_{n_k}.$$

Step 1 (Estimate for bound segments): Observe that

$$\frac{\Theta_{n_k}}{J^{n_k+p_k}(c_0)} = \frac{1}{J^{p_k}(c_{n_k}) d_C(c_{n_k}) d_S(c_{n_k})} + \sum_{i=n_k+1}^{n_k+p_k-1} \frac{1}{J^{n_k+p_k-i}(c_i) d_C(c_i) d_S(c_i)}.$$

To estimate the first term we use Lemma 2.5(b) to obtain

$$(12) \quad \frac{1}{J^{p_k}(c_{n_k})d_C(c_{n_k})d_S(c_{n_k})} \leq \sigma^{-1}(d_C(c_{n_k}))^{-\frac{16\alpha}{\lambda}}.$$

To estimate the second term we let \tilde{c} be the critical point to which c_{n_k} is bound. By using Lemma 2.2 and (4) in the proof of Lemma 2.2, which implies $d_C(c_i) > \frac{1}{2}d_C(\tilde{c}_{i-n_k-1})$ and $d_S(c_i) > \frac{1}{2}d_S(\tilde{c}_{i-n_k-1})$ for $i \in [n_k + 1, n_k + p_k - 1]$, we have

$$J^{n_k+p_k-i}(c_i)d_C(c_i)d_S(c_i) \geq K^{-1}J^{n_k+p_k-i}(\tilde{c}_{i-n_k-1})d_C(\tilde{c}_{i-n_k-1})d_S(\tilde{c}_{i-n_k-1}) \geq \sigma^2 L^{-5\alpha(i-n_k-1)}$$

where the last inequality is obtained by using (G1), Lemma 2.6(a) and (G3) for \tilde{c} . Summing this estimate over all i and combining the result with (12),

$$(13) \quad |(f^{n_k+p_k})'c_0|^{-1}\Theta_{n_k} \leq \sigma^{-1}(d_C(c_{n_k}))^{-\frac{16\alpha}{\lambda}} + \sigma^{-2}L^{6\alpha p_k} \leq (d_C(c_{n_k}))^{-\frac{18\alpha}{\lambda}}.$$

Here for the last inequality we use $\sigma \gg \delta$ and $L^{6\alpha p_k} \leq |d_C(c_{n_k})|^{-\frac{12\alpha}{\lambda}}$ from Lemma 2.5(a).

Step 2 (Estimate for free segments): By definition,

$$\frac{\Theta_0}{J^\nu(c_0)} = \sum_{i \in [0, \nu-1] \setminus (\cup [n_k, n_k+p-1])} \frac{1}{J^{\nu-i}(c_i)d_C(c_i)d_S(c_i)}.$$

Here we can not simply use (G3) for $d_S(c_i)$ in proving (15). We observe, instead, that either we have $v_i \notin S_\sigma$, for which $d_S(v_i) > \sigma$; or $c_i \in S_\sigma$ for which we have

$$J^{\nu-i}(c_i) = J^{\nu-i+1}(c_{i+1})J(c_i) \geq K^{-1}L^{\frac{\lambda}{3}(\nu-i+1)}(d_S(c_i))^{-1}.$$

It then follows, by using $d_C(c_i) > \delta$, that

$$(14) \quad \frac{\Theta_0}{J^\nu(c_0)} \leq \sum_{i \in [0, \nu-1] \setminus (\cup [n_k, n_k+p-1])} KL^{-\frac{\lambda}{3}(\nu-i)}(\sigma\delta)^{-1} \leq \frac{1}{\sigma\delta}.$$

This estimate is unfortunately not good enough for (15). To obtain (15), we need to use $C_{\delta^{\frac{1}{20}}}$ in the place of C_δ to define a new bound/free structure for each free segment out of C_δ . For the new free segments, we can now replace δ by $\delta^{\frac{1}{20}}$ in (14); for the bound segments, we use (13) with $d_C > \delta$. We then obtain

$$(15) \quad \frac{\Theta_0}{J^\nu(c_0)} \leq \frac{1}{\sigma\delta^{\frac{1}{20}}} + \sum_{i \in [0, \nu-1] \setminus (\cup [n_k, n_k+p-1])} KL^{-\frac{1}{3}\lambda(\nu-i)}\delta^{-\frac{18\alpha}{\lambda}} < \frac{1}{\delta^{\frac{1}{3}}}.$$

Step 3 (Proof of the Lemma): From the assumption that ν is a deep free return, we have

$$|d_C(c_{n_k})|^{-1} \leq |d_C(c_\nu)|^{-2} \prod_{j: n_j \in (n_k, \nu)} |d_C(c_{n_j})|^{-2}.$$

Substituting this into (13) gives

$$(16) \quad J^{n_k+p_k}(c_0)^{-1}\Theta_{n_k} \leq |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j: n_j \in (n_k, \nu)} |d_C(c_{n_j})|^{-\frac{36\alpha}{\lambda}}.$$

Meanwhile, splitting the orbit from time $n_k + p_k + 1$ to ν into bound and free segments and we have

$$(17) \quad J^{\nu-n_k-p_k}(c_{n_k+p_k})^{-1} \leq \left(\prod_{j: n_j \in (n_k, \nu)} J^{p_j}(c_{n_j}) \right)^{-1}.$$

Multiplying (16) with (17) gives

$$\begin{aligned} J^\nu(c_0)^{-1} \Theta_{n_k} &\leq |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j: n_j \in (n_k, \nu)} \left(J^{p_j}(c_{n_j}) \cdot |d_C(c_{n_j})|^{\frac{36\alpha}{\lambda}} \right)^{-1} \\ &\leq |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j: n_j \in (n_k, \nu)} (d_C(c_{n_j}))^{\frac{1}{2}} \leq \delta^{(t-k)/2} |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}}. \end{aligned}$$

where for the second inequality we use Lemma 2.5(b) for $J^{p_j}(c_{n_j})$, and for the last we use $d_C(c_{n_j}) < \delta$. Thus

$$\sum_{n_k \in [0, \nu-1]} J^\nu(c_0)^{-1} \Theta_{n_k} \leq |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}} \sum_{k=1}^t \delta^{(t-k)/2} \leq 2 |d_C(c_\nu)|^{-\frac{36\alpha}{\lambda}}.$$

Combining this with (15) we obtain

$$J^\nu(c_0)^{-1} D_\nu^{-1} = \sqrt{L} \left(\sum_{1 \leq k \leq t} J^\nu(c_0)^{-1} \Theta_{n_k} + J^\nu(c_0)^{-1} \Theta_0 \right) \leq \frac{1}{\sqrt{d_C(c_\nu)}}.$$

This completes the proof of Lemma 2.9. \square

3. MEASURE OF THE SET OF EXCLUDED PARAMETERS

The rest of the proof of the theorem goes as follows. For $n > N_0$, let

$$\Delta_n = \{a \in \Delta_N : (\text{R1})_n \text{ and } (\text{G3})_n \text{ hold}\},$$

and set $\Delta = \bigcap_{n \geq N} \Delta_n$. This is our parameter set in the theorem. In this section we show that Δ has positive Lebesgue measure, and $|\Delta| \rightarrow 1$ as $L \rightarrow \infty$. To this end, define two parameter sets as follows:

$$E_n = \{a \in \Delta_{n-1} \setminus \Delta_n : (\text{R})_n \text{ fails for } f_a\};$$

$$E'_n = \{a \in \Delta_{n-1} \setminus \Delta_n : (\text{G3})_n \text{ fails for } f_a\}.$$

Obviously, $\Delta_{n-1} \setminus \Delta_n \subset E_n \cup E'_n$. We show that the measures of these two sets decrease exponentially fast in n . Building on preliminary results in Sect.3.1, in Sect.3.2 we estimate the measure of E_n . In Sect.3.3 we estimate the measure of E'_n and complete the proof of the theorem.

Remark 3.1. The following will be used in the argument: Let $a \in \Delta_{n-1}$. By the definition of Δ_{n-1} , $(\text{R})_{n-1}$ holds, and thus by Lemma 2.8, $(\text{G1})_{n-1}$ $(\text{G2})_{n-1}$ hold.

3.1. Equivalence of derivatives and distortion in parameter space. For $c \in C$ and $i \geq 0$, we define $\gamma_i^{(c)} : \Delta_N \rightarrow S^1$ by letting $\gamma_i^{(c)}(a) = f_a^{i+1}(c)$. In this subsection we denote $c_i(a) = \gamma_i^{(c)}(a)$ and $\tau_i(a) = \frac{dc_i(a)}{da}$.

Lemma 3.2. *Let $a \in \Delta_{n-1}$. Then, for all $c \in C$ and $k \leq n$ we have*

$$\frac{1}{2} \leq \frac{|\tau_k(a)|}{|(f_a^k)'c_0(a)|} \leq 2.$$

Proof. We have

$$(18) \quad \tau_k(a) = 1 + f_a' c_{k-1}(a) \cdot \tau_{k-1}(a).$$

Using this inductively and then dividing the result by $(f_a^k)'c_0(a)$, which is nonzero by $(G2)_{n-1}$ we obtain

$$\frac{\tau_k(a)}{(f_a^k)'c_0(a)} = 1 + \sum_{i=1}^k \frac{1}{(f_a^i)'c_0(a)}.$$

This lemma then follows from applying $(G2)_{n-1}$. \square

For $a_* \in [0, 1)$, $c \in C$ and $c_0 = fc$, define

$$I_n(a_*, c) = [a_* - D_n(c_0), a_* + D_n(c_0)]$$

where $D_n(c_0)$ is the same as the one in (2) with $f = f_{a_*}$.

Lemma 3.3. *Let $a_* \in \Delta_{n-1}$. For all $c \in C$, $a \in I_n(a_*, c)$ and $k \leq n$ we have*

$$\frac{1}{2} < \frac{|\tau_k(a)|}{|\tau_k(a_*)|} \leq 2.$$

Proof. Let $k \leq n$. To prove this lemma we inductively assume that, for all $j < k$,

$$(19) \quad \frac{|\tau_j(a)|}{|\tau_j(a_*)|} \leq 2, \quad \text{for all } a \in I_n(a_*, c).$$

We then prove the same estimate for $j = k$. Write I_n for $I_n(a_*, c)$. For all $a \in I_n$ we have

$$\begin{aligned} & \left| \log \frac{|\tau_{j+1}(a)|}{|\tau_{j+1}(a_*)|} - \log \frac{|\tau_j(a)|}{|\tau_j(a_*)|} \right| \leq \left| \log \frac{|\tau_{j+1}(a)|}{|\tau_j(a)|} - \log |f_{a_*}' c_j(a_*)| \right| \\ & + \left| \log \frac{|\tau_{j+1}(a_*)|}{|\tau_j(a_*)|} - \log |f_{a_*}' c_j(a_*)| \right| \leq (I)_a + (I)_{a_*} + (II), \end{aligned}$$

where

$$(I)_a = \left| \log \frac{\tau_{j+1}(a)}{\tau_j(a)} - \log |(f_a)' c_j(a)| \right|, \quad (II) = \left| \log \frac{(f_a)' c_j(a)}{(f_{a_*})' c_j(a_*)} \right|.$$

We claim that

$$(20) \quad |(II)| \leq 2L^{-\frac{1}{3}} \cdot d_j^{-1} \left[\sum_{i=0}^{n-1} d_i^{-1} \right]^{-1},$$

where d_i is the same as the one in (2) with $f = f_{a_*}$.

To prove (20), first we use (19) and Lemma 3.2 to obtain

$$(21) \quad |\gamma_j^{(c)}(I_n)| \leq 2|\tau_j(a_*)||I_n| \leq 4|(f_{a_*}^j)'c_0(a_*)||I_n| \leq 2L^{-\frac{1}{2}}d_C(c_j(a_*))d_S(c_j(a_*)).$$

This implies $d_S(c_j(a)) \geq \frac{1}{2}d_S(c_j(a_*))$ for all $a \in I_n$. Thus from Lemma 2.1(b),

$$|f''| \leq \frac{KL}{d_S(c_j(a_*))^2} \quad \text{on } \gamma_j^{(c)}(I_n).$$

It then follows that

$$|f'_a c_j(a) - f'_{a_*} c_j(a_*)| = |f'_{a_*} c_j(a) - f'_{a_*} c_j(a_*)| \leq \frac{KL}{d_S(c_j(a_*))^2} |c_j(I_n)| \leq \frac{KL}{d_S(c_j(a_*))^2} |\tau_j(a_*)| \cdot |I_n|,$$

where (19) is again used for the last inequality. We have then

$$|f'_a c_j(a) - f'_{a_*} c_j(a_*)| \leq \frac{KL^{\frac{1}{2}} d_C(c_j(a_*))}{d_S(c_j(a_*))} d_j^{-1} \left(\sum_{i=0}^{n-1} d_i^{-1} \right)^{-1} \leq L^{-\frac{1}{3}} |(f'_{a_*} c_j(a_*))| d_j^{-1} \left(\sum_{i=0}^{n-1} d_i^{-1} \right)^{-1},$$

where for the last inequality we used Lemma 2.1(a). (20) follows directly from the last estimate.

As for $(I)_a$, we have from (18),

$$(22) \quad (I)_a \leq \log \left(1 + \frac{1}{|(f_a)' c_j(a)| \cdot |\tau_j(a)|} \right) < \frac{1}{|(f_a)' c_j(a)| \cdot |\tau_j(a)|} < L^{-\frac{\lambda}{3}j}$$

where for the last estimates we use the inductive assumption (19) and Lemma 3.2 for $|\tau_j(a_*)|$. We also use

$$|(f_a)' c_j(a)| \geq \frac{1}{2} |(f_{a_*})' c_j(a_*)| \geq \frac{L}{2} \cdot \min\{\sigma, L^{-\alpha j}\}$$

where the second inequality follows from (G1). Then $\frac{|\tau_k(a)|}{|\tau_k(a_*)|} \leq 2$ now follows from combining (22) and (20) for all $j < k$. \square

3.2. Exclusion on account of (R). To estimate the measure of parameters excluded due to (R)_n, we divide the critical orbit $\{v_i, i \in [0, n]\}$ into free/bound segments for $a \in \Delta_{n-1}$, $c \in C$, and let $t_1 < t_2 < \dots < t_q \leq n$ be the consecutive times for *deep free returns* to C_δ . Let $c^{(i)}$ be the corresponding binding critical point at time t_i , r_i is the unique integer such that $|c_i - f^{\nu_i} x_*| \in (L^{-r_i}, L^{-r_i+1}]$. We call

$$\mathbf{i} := (t_1, r^{(1)}, c^{(1)}; t_2, r^{(2)}, c^{(2)}; \dots; t_q, r^{(q)}, c^{(q)})$$

the *itinerary* of $v_0 = f_a(c)$ up to time n .

Let $E_n(c, \mathbf{i})$ denote the set of all $a \in E_n$ for which (R)_{n,c} fails at time n , and the itinerary for $c_0(a) = f_a c$ up to time n is \mathbf{i} . To estimate the measure of E_n , we first estimate the measure of $E_n(c, \mathbf{i})$. We then combine it with a bound on the number of all feasible itineraries.

Lemma 3.4. $|E_n(c, \mathbf{i})| \leq L^{-\frac{1}{3}R}$, where $R = r_1 + r_2 + \dots + r_q$.

Proof. First of all, for $a_* \in E_n(c, \mathbf{i})$ and $1 \leq k \leq q$ we define a parameter interval $I_{t_k}(a_*)$ as follows. If $|\gamma_{t_k}^{(c)}(I_n(a_*, c))| < \frac{1}{4}$, then we let $I_{t_k}(a_*) = I_{t_k}(a_*, c)$. Otherwise, take $I_{t_k}(a_*)$ to be the interval of length $\frac{1}{10|\gamma_{t_k}^{(c)}(I_{t_k}(a_*, c))|} |I_{t_k}(a_*, c)|$ centered at a_* .

The proof of Lemma 3.4 is outlined as follows. For a compact interval I centered at a and $r > 0$, let $r \cdot I$ denote the interval of length $r|I|$ centered at a . For each $k \in [1, q]$, we choose a countable subset $\{a_{k,i}\}_i$ of $E_n(c, \mathbf{i})$ with the following properties:

- (i) the intervals $\{I_{t_k}(a_{k,i})\}_i$ are pairwise disjoint and $E_n(c, \mathbf{i}) \subset \bigcup_i L^{-r_k/3} \cdot I_{t_k}(a_{k,i})$;
- (ii) for each $k \in [2, q]$ and $a_{k,i}$ there exists $a_{k-1,j}$ such that $I_{t_k}(a_{k,i}) \subset 2L^{-r_{k-1}/3} \cdot I_{t_{k-1}}(a_{k-1,j})$.

Observe that the desired estimate follows from this.

For the definition of such a subset we need two combinatorial statements. The following elementary fact from Lemma 2.9 is used in the proofs of these two sublemmas: if $a \in E_n(c, \mathbf{i})$, then $(\gamma_{t_k}^{(c)}|I_{t_k}(a))^{-1}(c^{(k)})$ consists of a single point and is contained in $L^{-r_k/3} \cdot I_{t_k}(a)$.

Sublemma 3.5. *If $a, a' \in E_n(c, \mathbf{i})$ and $a' \notin I_{t_k}(a)$, then $I_{t_k}(a) \cap I_{t_k}(a') = \emptyset$.*

Proof. Suppose $I_{t_k}(a) \cap I_{t_k}(a') \neq \emptyset$. Lemma 2.2 gives $|I_{t_k}(a)| \approx |I_{t_k}(a')|$. This and $a' \in I_{t_k}(a)$ imply $(\gamma_{t_k}^{(c)}|I_{t_k}(a))^{-1}(c^{(k)}) \neq (\gamma_{t_k}^{(c)}|I_{t_k}(a'))^{-1}(c^{(k)})$. On the other hand, by the definition of the intervals $I_{t_k}(\cdot)$, $\gamma_{t_k}^{(c)}$ is injective on $I_{t_k}(a) \cup I_{t_k}(a')$. A contradiction arises. \square

Sublemma 3.6. *If $a, a' \in E_n(c, \mathbf{i})$ and $a' \in L^{-r_k/3} \cdot I_k(a)$, then $I_{t_{k+1}}(a') \subset 2L^{-r_k/3} \cdot I_{t_k}(a)$.*

Proof. We have $(\gamma_{t_k}^{(c)}|I_{t_k}(a))^{-1}(c^{(k)}) \notin I_{t_{k+1}}(a')$, for otherwise the distortion of $\gamma_{t_{k+1}}^{(c)}$ on $I_{t_{k+1}}(a')$ is unbounded. This and the assumption together imply that one of the connected components of $I_{t_{k+1}}(a') - \{a'\}$ is contained in $L^{-r_k/3} \cdot I_{t_k}(a)$. This implies the inclusion. \square

We are in position to choose subsets $\{a_{k,i}\}_i$ satisfying (i) (ii). Lemma 3.5 with $k = 1$ allows us to pick a subset $\{a_{1,i}\}$ such that the corresponding intervals $\{I_{t_1}(a_{1,i})\}$ are pairwise disjoint, and altogether cover $E_n(c, \mathbf{i})$. Indeed, pick an arbitrary $a_{1,1}$. If $I_{t_1}(a_{1,1})$ covers $E_n(c, \mathbf{i})$, then the claim holds. Otherwise, pick $a_{1,2} \in E_n(c, \mathbf{i}) - I_{t_1}(a_{1,1})$. By Lemma 3.5, $I_{t_1}(a_{1,1}), I_{t_1}(a_{1,2})$ are disjoint. Repeat this. By Lemma 3.5, we end up with a countable number of pairwise disjoint intervals. To check the inclusion in (i), let $a \in I_1(a_{1,i}) - L^{-r_1/3} \cdot I_1(a_{1,i})$. By Lemma 2.9, $|f_a^{t_1+1}c - c^{(1)}| \gg L^{-r_1}$ holds. Hence $x \notin E_n(c, \mathbf{i})$.

Given $\{a_{k-1,j}\}_j$, we choose $\{a_{k,i}\}_i$ as follows. For each $a_{k-1,j}$, similarly to the previous paragraph it is possible to choose parameters $\{a_m\}_m$ in $E_n(c, \mathbf{i}) \cap L^{-r_{k-1}/3} \cdot I_{t_{k-1}}(a_{k-1,j})$ such that the corresponding intervals $\{I_{t_k}(a_m)\}_m$ are pairwise disjoint and altogether cover $E_n(c, \mathbf{i}) \cap L^{-r_{k-1}/3} \cdot I_{t_{k-1}}(a_{k-1,j})$. In addition, Lemma 3.6 gives $\bigcup_m I_{t_k}(a_m) \subset 2L^{-r_{k-1}/3} \cdot I_{t_{k-1}}(a_{k-1,j})$. Let $\{a_{k,i}\}_i = \bigcup_j \{a_m\}$. This finishes the proof of Lemma 3.4. \square

Lemma 3.7. *Assume that f_a is such that $a \in \Delta_N$. Then the lengths of any given bound period due to a return to C_δ is $\geq \frac{1}{2}\alpha N$.*

Proof. From $a \in \Delta_N$ and Lemma 2.1(a) we have $J^{[\frac{1}{2}\alpha N]}(c_0) \leq (\sigma^{-1}L)^{\frac{1}{2}\alpha N}$ where $\sigma = L^{-\frac{1}{6}}$. It then follows that

$$D_{[\frac{1}{2}\alpha N]}(c_0) \geq \frac{L^{-\frac{1}{2}\sigma^2}}{J^{[\frac{1}{2}\alpha N]}(c_0)} \geq L^{-\frac{1}{2}\sigma^2}(L^{-1}\sigma)^{\frac{1}{2}\alpha N} \gg \delta.$$

This implies the lemma. \square

Observe that $E_n = \bigcup E_n(c, \mathbf{i})$, where the union runs over all $c \in C$ and all feasible itineraries $\mathbf{i} = (t_1, r^{(1)}, c^{(1)}; \dots; t_q, r^{(q)}, c^{(q)})$. By Lemma 3.7 we have $q \leq \frac{2n}{\alpha N}$. We also have $R = r_1 + r_2 + \dots + r_q > \frac{\lambda \alpha n}{20}$. The number of choices for (r_1, \dots, r_q) satisfying $r_1 + \dots + r_q = R$ is $\binom{R+q}{q}$, and the possible number of choices for the deep free return times is $\binom{n}{q}$. By Stirling's formula for factorials, $\binom{n}{q} \leq e^{\beta(N)n}$ and $\binom{R+q}{q} \leq e^{\beta(N)R}$, where $\beta(N) \rightarrow 0$ as $N \rightarrow \infty$. Using these and Lemma 3.4 we conclude that

$$(23) \quad |E_n| \leq \sum_{1 \leq q \leq \frac{2n}{\alpha N}} \sum_{R > \frac{\lambda \alpha n}{20}} (\#C)^q \binom{n}{q} \binom{R+q}{q} L^{-\frac{R}{3}} \leq L^{-\frac{\lambda \alpha n}{100}},$$

where the last inequality holds for sufficiently large N .

3.3. Exclusion on account of (G3). Estimates for exclusions due to $(G3)_n$ are much simpler.

Lemma 3.8. *For $a_* \in \Delta_{n-1}$, $c \in C$, $|\gamma_n^{(c)}(I_n(a_*, c))| \geq L^{-3\alpha n}$.*

Proof. Write $f = f_{a_*}$, $c_i = f^{i+1}c$. Since $a_* \in \Delta_{n-1}$, $(G1)_{n-1}$ and $(G2)_{n-1}$ hold for c_0 by Lemma 2.8. Using Lemma 3.2 and 3.3, we have

$$|\gamma_n^{(c)}(I_n(a_*, c))| \geq 4J^n(c_0) \cdot |I_n(a_*, c)| \geq \frac{1}{\sqrt{L}} \left(\sum_{i=0}^{n-1} (J^n(c_0)d_i)^{-1} \right)^{-1}.$$

If $c_i \notin S_\sigma$, then

$$J^n(c_0)d_i = J^{n-i}(c_i)d_C(c_i)d_S(c_i) \geq KL^{-2\alpha i}\sigma.$$

If $c_i \in S_\sigma$, then $d_S(c_i) \geq |f'c_i|^{-1}$ from Lemma 2.1(a), and we have

$$J^n(c_0)d_i = J^{n-i}(c_i)d_C(c_i)d_S(c_i) \geq J^{n-i-1}(c_{i+1})d_C(c_i) \geq L^{-\alpha(i+1)}\sigma$$

where $(G1)_{n-1}$ is used for the last inequality. Hence we obtain

$$|\gamma_n^{(c)}(I_n(a_*, c))| > \frac{1}{\sqrt{L}} \left(\sum_{i=0}^{n-1} KL^{2\alpha i}\sigma \right)^{-1} \geq L^{-3\alpha n}.$$

□

We now estimate the measure of E'_n . For $c \in C$ and $s \in S$, let $E'_n(c, s)$ denote the set of all $a \in E'_n$ such that $d(\gamma_n^{(c)}(a), s) < L^{-4\alpha n}$. For $a \in E'_n(c, s)$, define an interval $I_n(a)$ centered at a , similarly to the definition of $I_{t_k}(\cdot)$ in the beginning of the proof of Lemma 3.4 (replace t_k by n). We claim that:

- if $a \in E'_n(c, s)$, then $I_n(a) \setminus L^{-\alpha n/2} \cdot I_n(a)$ does not intersect $E'_n(c, s)$;
- if $a, \tilde{a} \in E'_n(c, s)$ and $\tilde{a} \notin I_n(a)$, then $I_n(a) \cap I_n(\tilde{a}) = \emptyset$.

The first item follows from Lemma 3.8 and Lemma 3.3. The second follows from the injectivity argument used in the proof of Lemma 3.6. It follows that $|E'_n(c, s)| \leq L^{-\frac{1}{2}\alpha n}$, and therefore

$$(24) \quad |E'_n| < \#C \#S \cdot L^{-\frac{1}{2}\alpha n} < L^{-\frac{1}{3}\alpha n}.$$

(23), (24) and Lemma 2.3 altogether yield $|\Delta| \rightarrow 1$ as $L \rightarrow \infty$.

APPENDIX A. PROOFS OF LEMMAS 2.3 AND 2.4

In this appendix we prove Lemmas 2.3 and 2.4. Since various ideas and technical tools developed in the main text are needed, the correct order for the reader is to go over the main text first before getting into the details of these two proofs.

A.1. Proof of Lemma 2.3. Let $f = f_{a_*}$ and $c \in C$. Let $I_n(a_*, c) = [a_* - D_n(c_0), a_* + D_n(c_0)]$, where $D_n(c_0)$ is the same as the one in (2) with $f = f_{a_*}$. Let $\gamma_n^{(c)}(a) = f_a^{n+1}c$ and $\tau_n^{(c)}(a) = \frac{d}{da}\gamma_n^{(c)}(a)$.

Lemma A.1. *Let $1 \leq n \leq N_0$, $f = f_{a_*}$, $a_* \in \Delta_{n-1}$ and let $c \in C$. We have:*

- (a) $J^{j-i}(c_i) \geq (K^{-1}L\sigma)^{j-i}$ for all $0 \leq i < j \leq n$;
- (b) $J^n(x) \leq 2J^n(y)$ for all $x, y \in [c_0 - D_n(c_0), c_0 + D_n(c_0)]$;
- (c) $\frac{1}{2} \leq \frac{|\tau_n^{(c)}(a_*)|}{J^n(c_0)} \leq 2$;
- (d) $\frac{1}{2} \leq \frac{|\tau_n^{(c)}(a)|}{|\tau_n^{(c)}(a_*)|} \leq 2$ for all $a \in I_n(a_*, c)$;
- (e) $|\gamma_n^{(c)}(I_n(a_*, c))| \geq L^{1/7}\sigma$.

Proof. Item (a) follows directly from Lemma 2.1. (b) is in Lemma 2.2, (c) is in Lemma 3.2 and (d) is in Lemma 3.3. As for (e), let $i \in [0, n-1]$. (a) gives

$$J^n(c_0)d_i(c_0) = J^{n-i}(c_i)d_C(c_i)d_S(c_i) \geq (K_0^{-1}L\sigma)^{n-i}\sigma^2,$$

where $d_i(c_0)$ is the same as the one in (2) with $f = f_{a_*}$. Taking reciprocals and then summing the result over all $i \in [0, n-1]$ we have

$$\sum_{i=0}^{n-1} J^n(c_0)^{-1}d_i(c_0)^{-1} \leq \frac{K_0}{L\sigma^3}.$$

Recall that $\sigma = L^{-\frac{1}{6}}$. We have

$$|\gamma_n^{(c)}(I_n(a_*, c))|^{-1} \leq KJ^n(c_0)^{-1}D_n^{-1}(c_0) = K\sqrt{L} \cdot \sum_{i=0}^{n-1} J^n(c_0)^{-1}d_i(c_0)^{-1} \leq \frac{1}{L^{\frac{1}{7}}\sigma},$$

where the first inequality follows from (c) and (d). \square

For $c \in C$, $s \in C \cup S$, let $E_n(c, s)$ denote the set of all $a \in \Delta_{n-1} \setminus \Delta_n$ so that $d(\gamma_n^{(c)}(a), s) \leq \sigma$. For $a \in E_n(c, s)$, define a parameter interval $I_n(a)$ as follows. If $|\gamma_n^{(c)}(I_n(a, c))| < \frac{1}{4}$, then we let $I_n(a) = I_n(a, c)$. Otherwise, define $I_n(a)$ to be the interval of length $\frac{1}{10|\gamma_n^{(c)}(I_n(a, c))|}|I_n(a, c)|$ centered at a . We claim that:

- if $a \in E_n(c, s)$, then $I_n(a) \setminus L^{-1/8} \cdot I_n(a)$ does not intersect $E_n(c, s)$;
- if $a, \tilde{a} \in E_n(c, s)$ and $\tilde{a} \notin I_n(a)$, then $I_n(a) \cap I_n(\tilde{a}) = \emptyset$.

The first item follows from Lemma A.1(c)-(e). The second follows from the fact that the map $\gamma_n^{(c)}$ is injective on $I_n(a)$. Observe that from this we have $|E_n(c, s)| < L^{-\frac{1}{8}}$, and it follows that $|\Delta_{n-1} \setminus \Delta_n| < \#C(\#C + \#S)L^{-\frac{1}{8}}$, and we obtain

$$|\Delta_N| \geq 1 - \sum_{n=1}^N |\Delta_{n-1} \setminus \Delta_n| \geq 1 - L^{-\frac{1}{9}}.$$

This completes the proof of Lemma 2.3. \square

A.2. Proof of Lemma 2.4. We need the materials in Sect.2.4 and the definition of the bound/free structure in Sect.2.5.

Let $\delta_0 = L^{-\frac{1}{12}}$. For the orbit of x lying out of C_δ we first introduce the bound/free structure of Sect.2.5 by using C_{δ_0} in the place of C_δ . We then prove a result that is similar to Lemma 2.5, from which Lemma 2.4 would follow directly. To make this argument work, we first need to show that the bound periods on $C_{\delta_0} \setminus C_\delta$, as defined in Sect.2.4, are $\leq N$. Let $I_p(c)$ and the bound period be defined the same as in Sect.2.4.

Lemma A.2. *If $f = f_a$ is such that $a \in \Delta_N$, then for each $c \in C$ we have $[c - \delta_0, c - \delta] \cup [c + \delta, c + \delta_0] \subset \bigcup_{1 \leq p \leq N} I_p(c)$.*

Proof. Let $c_0 = fc$. It suffices to show that

$$(25) \quad L^{-1}D_N(c_0) < \delta^2 < \delta_0^2 < L^{-1}D_1(c_0)$$

where $D_n(c_0)$ is as in (2). Since $a \in \Delta_N$ we have $J^{N-1}(c_0) \geq (KL\sigma)^{N-1}$ from Lemma 2.1. It then follows that

$$D_N(c_0) \leq d_{N-1}(c_0) \leq (KL\sigma)^{-N+1} < L\delta^2.$$

On the other hand we have

$$D_1(c_0) = \frac{1}{\sqrt{L}} \cdot d_C(c_0)d_S(c_0) \geq L^{-\frac{1}{2}}\sigma^2,$$

from which the last inequality of (25) follows directly. \square

With the help of Lemma A.2, we know that the bound/free structure for the orbit of x out of C_δ is well-defined.

Lemma A.3. *Let $p \geq 2$ be the bound period for $y \in C_{\delta_0} \setminus C_\delta$. Then:*

- (a) *for $i \in [1, p]$, we have $d_C(f^i y) > \delta_0$ and $|(f^{i-1})'(c_0)|D_i(c_0) \geq L^{-\frac{1}{2}}\sigma^2$;*
- (b) *$J^p(y) \geq K^{-1}L^{-\frac{1}{2}}\sigma^2|c - y|^{-1} \geq K^{-1}L^{-\frac{1}{2}}\sigma^2(K_0^{-1}L\sigma)^{p-2}$;*
- (c) *$J^p(y) \geq L^{\frac{1}{300}p}$.*

Proof. (a) is a version of Sublemma 2.6. The estimates are better because here we have $f^{i-j}(c_j) > (KL\sigma)^{i-j}$, and $d_C(c_i), d_S(c_i) > \sigma$. As for (b) we have

$$J^p(y) \geq K^{-1}L|c - y|J^{p-1}(c_0) \geq K^{-1}|c - y|^{-1}J^{p-1}(c_0)D_p(c_0),$$

where for the last inequality we use $D_p(c_0) \leq KL|c - y|^2$ by the definition of p . The first inequality of (b) then follows by using (a). The second inequality of (b) follows from

$$|c - y| \leq D_{p-1}(c_0) \leq \frac{d_{p-2}(c_0)}{\sqrt{L}} \leq \frac{(K_0^{-1}L\sigma)^{-p+2}}{\sqrt{L}}.$$

Here, last inequality is because $a \in \Delta_N$.

If $p \geq 10$, then (c) is much weaker than the second inequality of (b). If $p < 10$, then (c) follows from the first inequality of (b) and the fact that $|c - y|^{-1} \geq \delta_0^{-1} = L^{\frac{1}{12}}$. \square

We are in position to finish the proof of Lemma 2.4. If $f^n x$ is free (this includes the case $f^n x \in C_\delta$), then we use Lemma A.3(c) for bound segments and $|f'| > K^{-1}L^{\frac{1}{12}}$ for iterates in free segments, which are out of C_{δ_0} . This proves Lemma 2.4(b). If $f^n x$ is bound, then there is a drop of a factor $> \delta$ at the last free return that can not be recovered. In this case we need the factor δ in Lemma 2.4(a). \square

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